Symmetric Stair Preconditioning of Linear Systems for Parallel Trajectory Optimization

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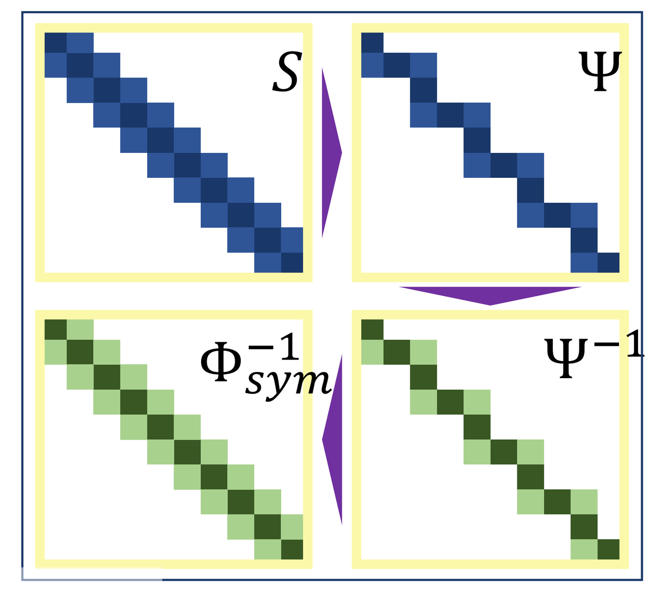
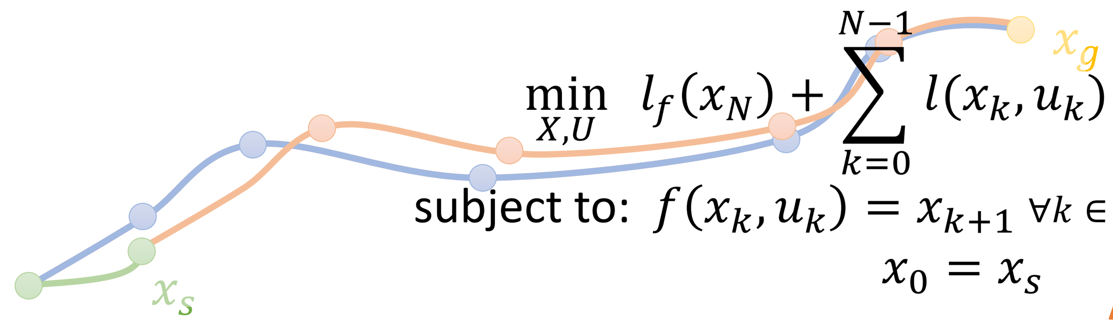
The Big Picture:

In this work, we present a **new parallel-friendly symmetric stair preconditioner** optimized for solving the linear systems underlying trajectory optimization problems. We prove that our preconditioner has **advantageous theoretical properties** when used with **iterative linear system solvers** such as a more clustered and bound Eigenspectrum. Numerical experiments with typical trajectory optimization problems reveal that our preconditioner provides up to a **34% reduction in condition number** and up to a **25% reduction in the number of linear system solver iterations**.



Direct Trajectory Optimization:

Trajectory optimization computes a robot's optimal path through an environment as a series of states, x, and controls, u, by minimizing a cost function, l, subject to discrete time dynamics, f.

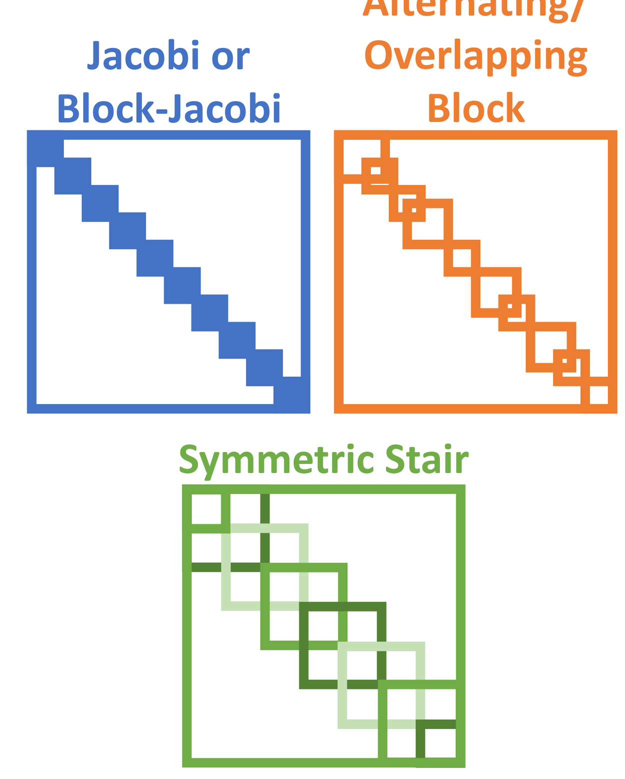


$$\min_{X,U} \; l_f(x_N) + \sum_{k=0}^{N-1} l(x_k, u_k)$$

$$\text{subject to: } f(x_k, u_k) = x_{k+1} \;\forall k \in [0, N) \qquad x_0 = x_s$$

The Need for Sparse Parallel Preconditioners:

One key computation for direct trajectory optimization problems is the repeated solving of the resulting Karush-Kuhn-Tucker linear system, which can be solved using the **symmetric positive definite and block tridiagonal *Schur Complement*, S**. Iterative linear system solves can accelerate such problems on parallel processors. However, these methods **require a preconditioner, $\Phi^{-1} \approx S^{-1}$**, as their convergence properties are related to the clustering and magnitude of the eigenvalues of $\Phi^{-1}S$. Furthermore, as the resulting algorithms leverage matrix-vector products with both Φ^{-1} and S, **efficient parallel solvers require Φ^{-1} to be both sparse and parallel friendly**. Previous works usually leverage Jacobi, Block-Jacobi, as well as Alternating/Overlapping Block preconditioners.



The Symmetric Stair Preconditioner:

The stair-splitting polynomial preconditioner for block-tridiagonal systems is parallel friendly to both compute and invert, with an **analytical inverse**, however it is not symmetric, preventing its use with standard iterative methods.

$$S = \begin{bmatrix} D_1 & O_1 & 0 \\ O_1^T & D_2 & O_2 \\ 0 & O_2^T & D_3 \end{bmatrix} = \Psi_l = \begin{bmatrix} D_1 & 0 & 0 \\ O_1^T & D_2 & O_2 \\ 0 & 0 & D_3 \end{bmatrix} - P_l = -\begin{bmatrix} 0 & O_1 & 0 \\ 0 & 0 & 0 \\ 0 & O_2^T & 0 \end{bmatrix} \;\blacktriangleright\; \Psi_l^{-1} = \begin{bmatrix} D_1^{-1} & 0 & 0 \\ -D_2^{-1}O_1^T D_1^{-1} & D_2^{-1} & -D_2^{-1}O_2 D_3^{-1} \\ 0 & 0 & D_3^{-1} \end{bmatrix}$$

$$\Psi_r = \begin{bmatrix} D_1 & O_1 & 0 \\ 0 & D_2 & 0 \\ 0 & O_2^T & D_3 \end{bmatrix} - P_r = -\begin{bmatrix} 0 & 0 & 0 \\ O_1^T & 0 & O_2 \\ 0 & 0 & 0 \end{bmatrix} \;\blacktriangleright\; \Psi_r^{-1} = \begin{bmatrix} D_1^{-1} & -D_1^{-1}O_1 D_2^{-1} & 0 \\ 0 & D_2^{-1} & 0 \\ 0 & -D_3^{-1}O_2^T D_2^{-1} & D_3^{-1} \end{bmatrix}.$$

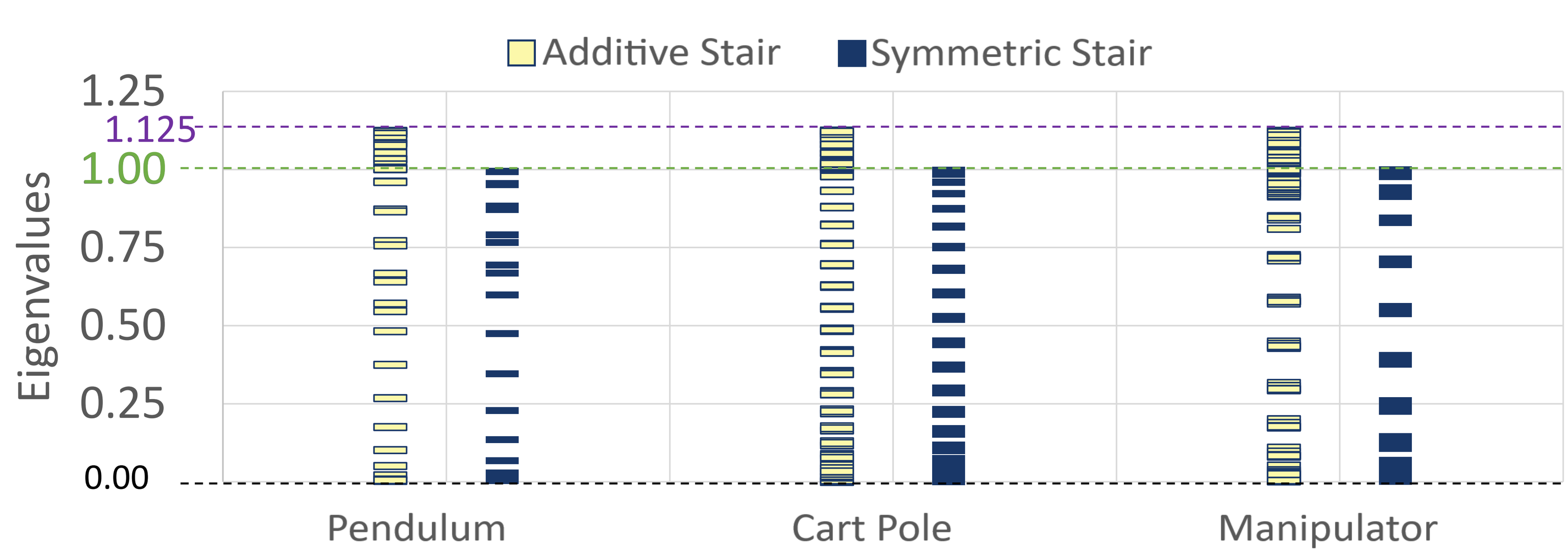
We prove that our symmetric stair preconditioner, Φ_{sym}^{-1}, provides an more clustered Eigenspectrum and bounds the spectral radius at one, unlike the existing symmetric stair-based preconditioner, the additive stair preconditioner, Φ_{add}^{-1}.

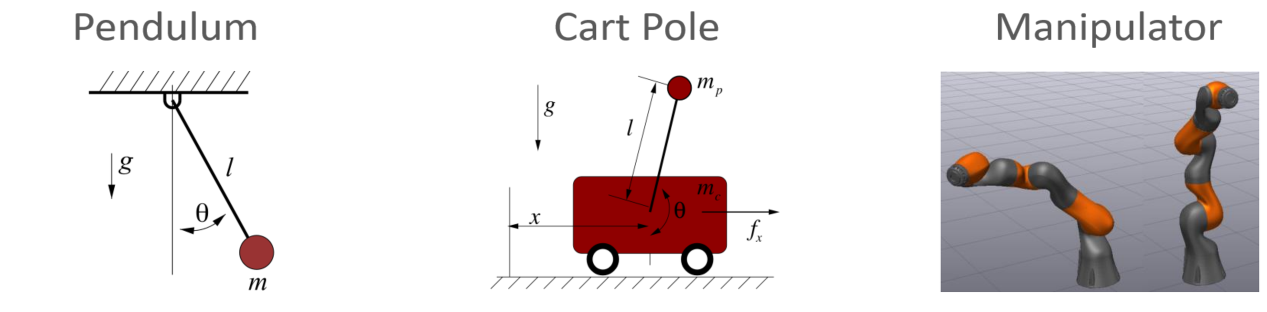
$$\Phi_{add}^{-1} = \frac{1}{2}(\Psi_l^{-1} + \Psi_r^{-1}) \;\blacktriangleright\; \lambda(\Phi_{add}^{-1}S) \in \left(0, \frac{9}{8}\right] \qquad \Phi_{sym}^{-1} = \Psi_l^{-1} + \Psi_r^{-1} - D^{-1} \;\blacktriangleright\; \lambda(\Phi_{sym}^{-1}S) \in (0, 1]$$

Experimental Results:

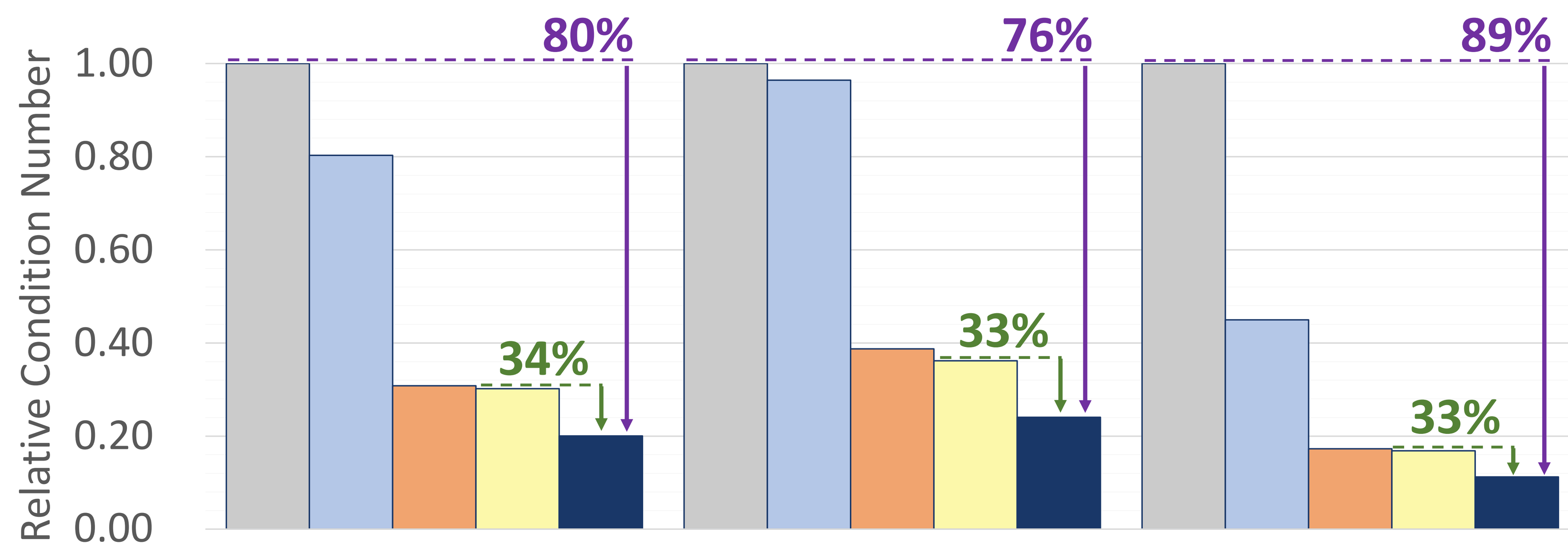
We provide numerical experiments comparing our preconditioner to previous parallel preconditioners for three canonical trajectory optimization problems: 1) a pendulum swing-up, 2) a cart-pole swing-up, and 3) a workspace motion for a 7-dof manipulator.

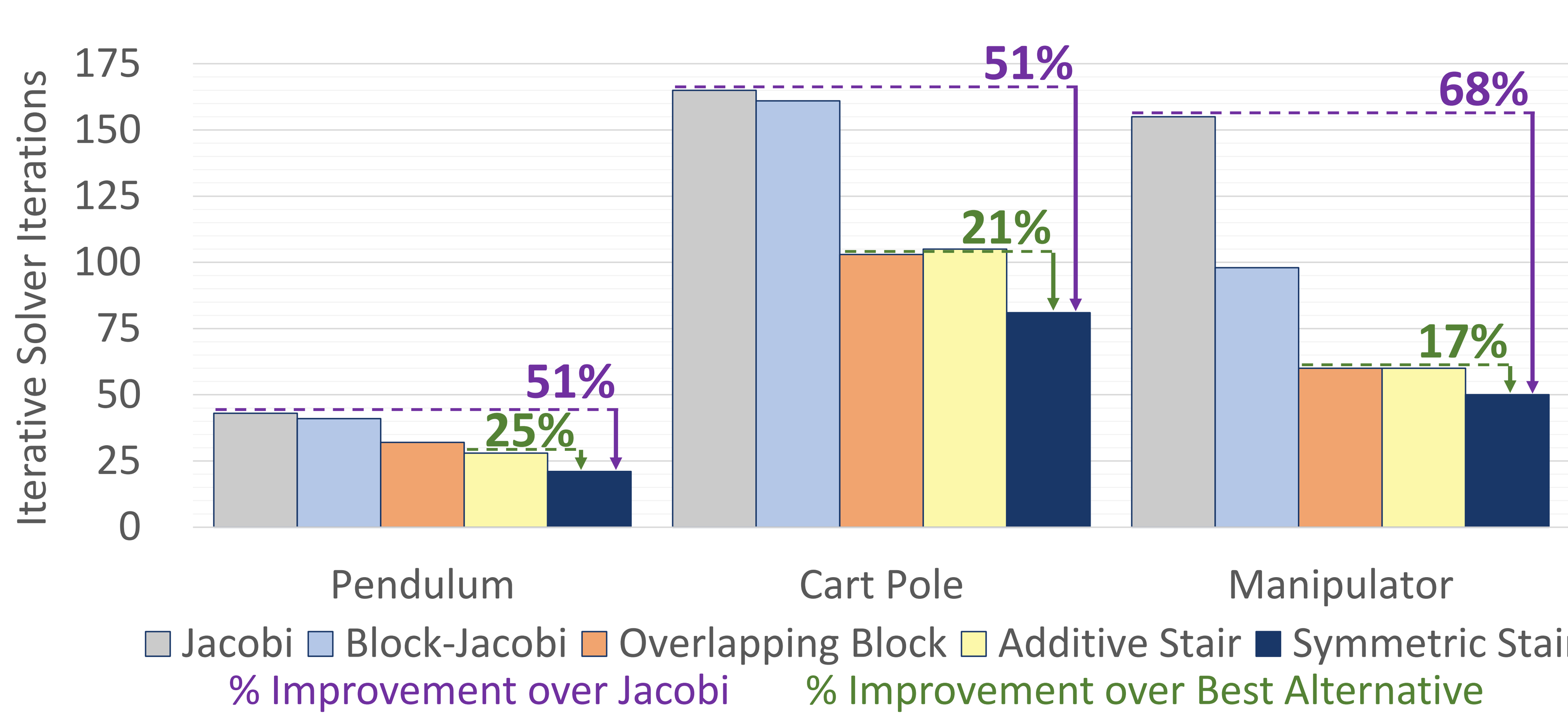
As shown in our proofs, our preconditioner provides a **clustered and bound Eigenspectrum.**





This both **reduces the condition number and, more importantly, the resulting iterations to convergence,** when used in conjunction with iterative linear system solvers.





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